Math 10B with Professor Stankova
Worksheet, Discussion #25; Tuesday, 4/24/2018
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1 Linear Systems of Differential Equations

1.1 Concepts

1. In order to solve a system of linear differential equations, we represent it in the form $\vec{y}' = A\vec{y}$. Then we find the eigenvalues of A, say λ_1, λ_2. If $\lambda_1 \neq \lambda_2$ are real, then we find the eigenvectors $\vec{v}_1, \vec{v}_2$ and the general solution is of the form $\vec{y} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda-2t}\vec{v}_2$.

1.2 Example

2. Find the general solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = y_1(t) + 4y_2(t) \\ y_2'(t) = 3y_2(t) \end{cases}$$

> **Solution:** Let $A = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(1-\lambda)(3-\lambda) = 0$ or $\lambda = 1, 3$. For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and for $\lambda = 3$, the eigenvector is given by $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Thus, the general solution is
>
> $$\vec{y} = c_1 e^t \vec{v}_1 + c_2 e^{3t}\vec{v}_2 = \begin{pmatrix} 4c_1 e^t + 4c_2 e^{3t} \\ 2c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + 2c_2 e^{3t} \\ c_2 e^{3t} \end{pmatrix}.$$

1.3 Problems

3. **TRUE** False If 2 is an eigenvalue for A, then 4 is an eigenvalue for A^2.

> **Solution:** Let $\vec{v}$ be the associated eigenvector so that $A\vec{v} = 2\vec{v}$. Then $A^2\vec{v} = A(A\vec{v}) = A(2\vec{v}) = 2^2\vec{v} = 4$ so A^2 has an eigenvalue 4 with eigenvector $\vec{v}$.

4. True **FALSE** If 2 is an eigenvalue of A and 3 is an eigenvalue of B, then $2 \cdot 3 = 6$ is an eigenvalue of AB.

5. True **FALSE** If two matrices A, B have the same eigenvalues, then they have the same solutions to $\vec{y}' = A\vec{y}$.

Solution: The general solution depends both on the eigenvalues and eigenvectors.

6. Find the solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = 5y_1(t) - 4y_2(t) \\ y_2'(t) = 4y_1(t) - 5y_2(t) \end{cases}$$

with $\vec{y}(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(5-\lambda)(-5-\lambda) + 16 = \lambda^2 - 9 = 0$ or $\lambda = -3, 3$. For $\lambda = -3$, the eigenvector is $\begin{pmatrix} -4 \\ -8 \end{pmatrix}$ and for $\lambda = 3$, the eigenvector is given by $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$. Thus, the general solution is

$$\vec{y} = c_1 e^{-3t}\vec{v}_1 + c_2 e^{3t}\vec{v}_2 = \begin{pmatrix} -4c_1 e^{-3t} - 4c_2 e^{3t} \\ -8c_1 e^{-3t} - 2c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-3t} + 2c_2 e^{3t} \\ 2c_1 e^{-3t} + c_2 e^{3t} \end{pmatrix}.$$

Now plugging in the initial conditions give $c_1 + 2c_2 = 3$ and $2c_1 + c_2 = 3$ so $c_1 = c_2 = 1$ and the solution is $\vec{y} = \begin{pmatrix} e^{-3t} + 2e^{3t} \\ 2e^{-3t} + e^{3t} \end{pmatrix}$.

7. Find the general solution to the systems of linear differential equations

$$\begin{cases} y_1'(t) = 2y_1(t) + y_2(t) \\ y_2'(t) = y_1(t) + 2y_2(t) \end{cases}$$

Solution: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then letting $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have that $\vec{y}' = A\vec{y}$. The eigenvalues of A are given by $(2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$ or $\lambda = 1, 3$. For $\lambda = 1$, the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and for $\lambda = 3$, the eigenvector is given by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the general solution is

$$\vec{y} = c_1 e^t \vec{v}_1 + c_2 e^{3t} \vec{v}_2 = \begin{pmatrix} c_1 e^t + c_2 e^{3t} \\ -c_1 e^t + c_2 e^{3t} \end{pmatrix}.$$

8. Verify that $\vec{x}(t) = \begin{pmatrix} 0 \\ -e^t \\ e^t \end{pmatrix}$, $\vec{y}(t) = \begin{pmatrix} e^{2t} \\ -2e^{2t} \\ 0 \end{pmatrix}$, $\vec{z}(t) = \begin{pmatrix} 0 \\ e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to $\vec{v}' = A\vec{v}$ where $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

Solution: Multiplying gives us $A\vec{x} = \vec{x}$, $A\vec{y} = 2\vec{y}$ and $A\vec{z} = 3\vec{z}$ which is what we wanted to show since $\vec{x}' = \vec{x}, \vec{y}' = 2\vec{y}, \vec{z}' = 3\vec{z}$.

9. Under the same notation as the previous problem. Write out the system of linear equations that $\vec{v}' = A\vec{v}$ represents and find the general solution.

Solution: It represents

$$\begin{cases} y_1'(t) = 2y_1(t) \\ y_2'(t) = 2y_2(t) + y_3(t) \\ y_3'(t) = 2y_1(t) + y_2(t) + 2y_3(t) \end{cases}.$$

The general solution is of the form $c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \begin{pmatrix} c_2 e^{2t} \\ -c_1 e^t - 2c_2 e^{2t} + c_3 e^{3t} \\ c_1 e^t + c_3 e^{3t} \end{pmatrix}$.

10. Still with the same notation, what are the eigenvalues and eigenvectors of A?

Solution: The eigenvalues are $1, 2, 3$ with eigenvectors $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ respectively.

2 Miscellaneous

2.1 Problems

11. Let $V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Let $A = V \cdot \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \cdot V^{-1}$. Calculate A.

Solution: $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$.

12. With the same A as above, calculate the eigenvalues and eigenvectors of A. What do you notice? How does this relate to V?

Solution: The eigenvalues of A are 3 and -1. For $\lambda = 3$, the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and for $\lambda = -1$, the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. These are exactly the elements that appear in the diagonal matrix in the center and the columns of V.

13. (Challenge) Create a matrix with eigenvalues $\lambda = 1, 2$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Solution: Let $V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then let $A = V \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. This has the desired eigenvalues and eigenvectors.